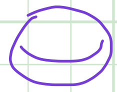
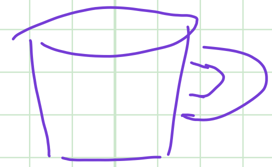
What is topology

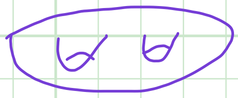
Mathematics of classifying "shapes" of various objects.

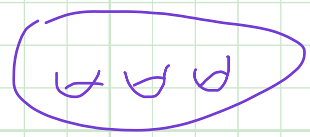
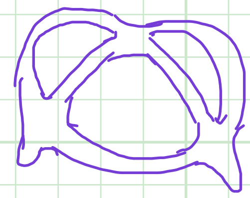
⇒ Example: classification of 2D manifolds (that can be embedded in 3D)

2D manifold is a 2D surface with no boundries, some examples are:

⇒ sphere 

⇒ doughnut / coffee cup  

⇒ two holed doughnut 

⇒ three holed doughnut / pretzel  

...

genus g = # of holes in a surface

manifolds with same g can be smoothly deformed into each other. Manifolds with different g can not, these must be to'n apart and re-glued.

⇒ For topological superconductivity we will need a different topology that of classifying "shapes" of functions. ⇒ Homotopy theory

⇒ Specifically we will treat our Hamiltonian as a function from k-space to the unit ring and thus classify the "shapes" of $H[k]$.

For detailed introduction to topology in Physics see N.D. Mermin RMP <u>51</u> 591 (1979)

Topology in "topological superconductors"

Last time, we studied the Kitaev model, which is encoded in the Hamiltonian:

$$H = -\sum_i \left(c_i^+ c_{i+1} + c_{i+1}^+ c_i\right) + \Delta \sum_i \left(c_i^+ c_{i+1}^+ + c_{i+1} c_i\right) - 2\mu \sum_i c_i^+ c_i$$

In momentum space, this Hamiltonian becomes:

$$H = \sum_k \left[-2t\cos(k)\, c_k^+ c_k - \Delta i \sin(k)\left(c_k^+ c_{-k}^+ + c_k c_{-k}\right) - \mu c_k^+ c_k\right]$$

As the pairing terms mix fermions with momentum k and $-k$, it is useful to rewrite the Hamiltonian using $k,-k$ pairs:

$$H = \sum_k \begin{pmatrix} c_k^+ & c_{-k} \end{pmatrix} \begin{pmatrix} -t\cos(k) - \mu & -i\Delta \sin(k) \\ i\Delta \sin(k) & t\cos(k) + \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$$

We can think of this 2×2 Hamiltonian as a Hamiltonian for an $S=\frac{1}{2}$ pseudospin. The z-direction of this pseudospin corresponds to the mixing of a particle-like $(S^z = +\frac{1}{2})$ and a hole-like $(S^z = -\frac{1}{2})$ state.

Generically, a Hamiltonian for a single spin ½ looks like

$$H = \underbrace{\vec{\sigma}}_{\text{spin direction}} \cdot \underbrace{\vec{B}}_{\text{magnetic field}}$$

where $\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. Indeed, any 2×2 Hamiltonian can be represented in the form

$$\mathcal{H} = \sigma^x B^x + \sigma^y B^y + \sigma^z B^z + \varepsilon \underbrace{\mathbb{1}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

In our case, that is for the Kitaev model, we have

$$\mathcal{H} = \sigma^y B^y + \sigma^z B^z$$

where

$$B^y = \Delta \sin(k) \quad \text{and} \quad B^z = -t\cos(k) + \mu$$

Now for some mathematics:

Let us consider the dependence of the 2x2 Hamiltonian on the momentum $H[k]$. As we move from one point to the next in k-space, the Hamiltonian changes continuously. A better way to think about this continuous change is in terms of the associated B-field. At each point in k-space we have a B-fied $B(k)$.

Q: what can we say about the function $B(k)$?

Let us make some assumptions:

(0) Let us assume that $B(k)$ is a continuous function

(1) let us assume that $|B(k)| > 0$ for all points in k-space

⇒ This assumption means that we can define a direction

$$n(k) = \frac{B(k)}{|B(k)|}$$ for each point in k-space.

⇒ Physically this assumption states that there is finite S.C. gap for all k points.

(2) Let us assume that $B(k) = B(k+2\pi)$, as demanded by Bloch theorem.

⇒ Mathematicians tell us that there are distinct classes of functions $B(k)$. Members of the same class can be continuously deformed into each other without violating the assumptions, e.g. via:

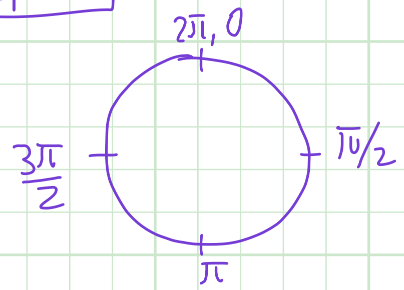
$$B(k,\tau) = B_1(k)\,\tau + B_2(k)(1-\tau)$$

while members of different classes cannot. [These classes are members of the fundamental group π_1]

⇒ Why are there these distinct classes of $B(k)$ functions?

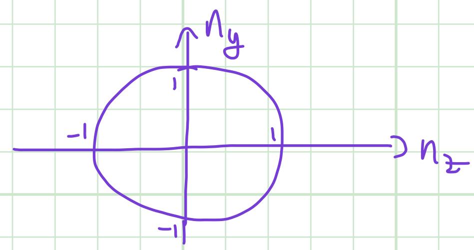
Instead of thinking about $B(k)$, let us focus on $n(k)$.

k-space



the map $n(k)$ connects these two spaces

n-space



Since k-space is periodic let us represent it as a ring.

Since B lives in the y-z plane so does n.

$\Rightarrow$ By assumption $|B|>0 \ \forall \ k$ hence we can define an $n(k)$ $\forall k$.

The different classes of $n(k)$ correspond to how many times $n(k)$ wraps around the ring in n-space as k goes from 0 to 2π.

$\Rightarrow$ Demonstrate with a rubber band

$\Rightarrow$ can't change class without either breaking + regluing the rubber band or going over "the north pole"

Topology in Superconductivity refers to distinct "classes" of superconductors that cannot be deformed into each other smoothly, i.e. while maintaining a finite SC gap.

The "modified" Kitaev model can be in one of two classes depending on the parameters t, Δ, μ. Instead of investigating this analytically, let us do this numerically.

Numerical investigation of topology of Kitaev model:

Starting with our definition for $B(k)$

$$B^y = \Delta \sin(k) \quad \text{and} \quad B^z = -t\cos(k) - \mu$$

we define $n(k)$ as:

$$\{n^y(k), n^z(k)\} = \frac{\{\Delta \sin(k), -t\cos(k) - \mu\}}{\sqrt{(t\cos(k)+\mu)^2 + (\Delta \sin(k))^2}}$$

Let us plot $\{n^y(k), n^z(k)\}$ as we vary k from 0 to 2π. WLOG we choose $t=1$. We let Δ and μ vary in the range $\Delta, \mu \in [-2 \dots 2]$.

$\Rightarrow$ Play with Mathematica

We observe that $n(k)$ changes qualitatively when $\mu = \pm 1$, independent of the value of Δ. Why?

Let us ask when does the gap in spectrum of H_{Kitaev} close?

This occurs when $|B| = 0$:

$$(t\cos(k)+\mu)^2 + (\Delta \sin k)^2 = 0$$

$\hookrightarrow$ $k = 0, \pi$ otherwise Δ term is finite

$\hookrightarrow$ $(\mu \pm t) = 0$ $\Rightarrow$ $\mu = \mp t$ are the critical points just as Mathematica told us.

What do these two distinct phases correspond to?

Kitaev's topological phase: smoothly connected to $\Delta = t = 1$ $\mu = 0$

$\Rightarrow$ eigenfermions live on bonds

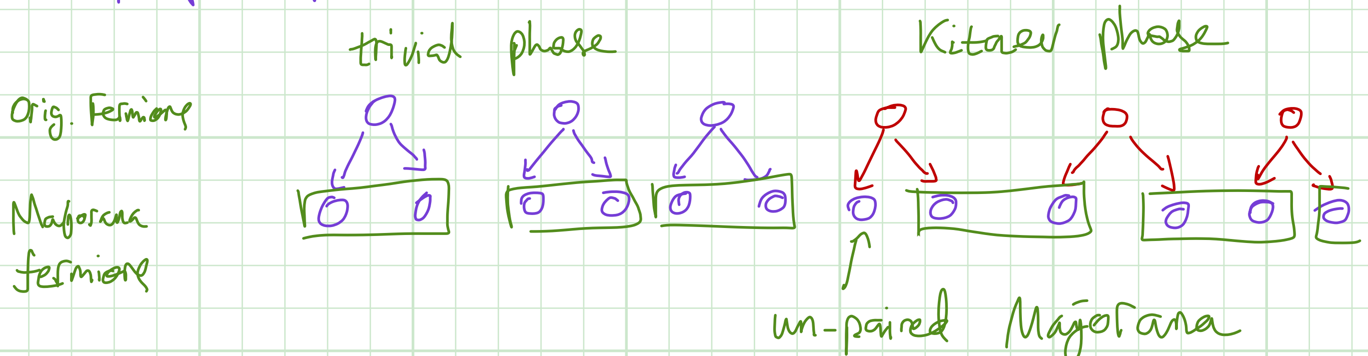
non-topological phase: smoothly connected to $\Delta = t = 0$ $\mu = 1$

$\Rightarrow$ eigenfermions are the original site fermions.

Why is topology important?

Index theorem tells us that at the interface between two topologically distinct superconductors we will always find a Majorana zero mode.

"Proof" of theorem:



What is non-locality and why do we care?

Consider a NT (non-topological) – T (topological) – NT device

This device hosts two Majorana zero modes at the two T-NT interfaces.

$\Rightarrow$ We can not affect the occupation # of the fermion $\gamma_1 + i\gamma_2$ by local perturbations $\Rightarrow$ ie. those near only one of the interfaces.

$\Rightarrow$ This is good for quantum information storage: the quantum info is simultaneously stored on two interfaces hence non-local storage.

Quantum bit with Majorana zero modes:

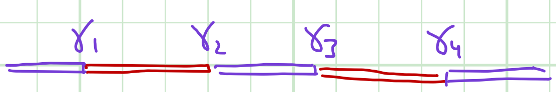
We want to store quantum information in the Majorana zero modes. We may want to store quantum info in the device above

$\Rightarrow$ define $f_{12}^{+} = \gamma_1 + i\gamma_2$

$$\Rightarrow \begin{cases} |vac\rangle = |0\rangle & [\text{qubit}] \\ f_{12}^{+}|vac\rangle = |1\rangle & [\text{qubit}] \end{cases}$$

$\Rightarrow$ problems: (1) we want to mix $|0\rangle$ and $|1\rangle$, which means we are mixing states of different parity. These mixtures are believed to be fragile. (2) Also, to drive $|0\rangle \to |1\rangle$ transition we need to change parity of the wave function.

$\Rightarrow$ Alternative: use another pair of Majorana zero modes as a "parity reservoir"



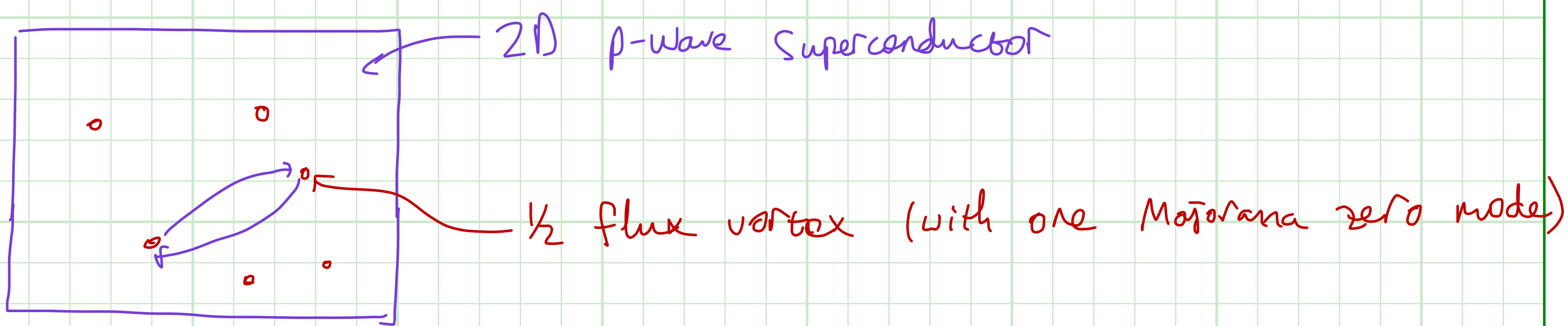
$f_{12}^{+} = \gamma_1 + i\gamma_2 \qquad f_{34}^{+} = \gamma_3 + i\gamma_4$

We can run qubit in Total Parity even or Total Parity odd Sector:

	Total Parity Even	Total Parity Odd
logical $\|0\rangle$	$\|vac\rangle$	$f_{12}^{+} f_{34}^{+} \|vac\rangle$
logical $\|1\rangle$	$f_{12}^{+} \|vac\rangle$	$f_{34}^{+} \|vac\rangle$

Topologically protected quantum computation + non-abelien statistics.

2D equivalent of the Kitaev wire is a p-wave superconductor. In this case the Majorana zero modes "live" inside half-flux vortices.



What is the number of ground states?

If we have n [k vortices]

$\Rightarrow n$ Majorana zero modes

$\Rightarrow n/2$ complex fermions with zero energy

$\Rightarrow 2^{n/2}$ ground states. $\Rightarrow$ This is the Hilbert space on which we can do quantum computation.

Braiding: we can go between different ground states by adiabatically (slowly swapping location of vortices).

$\Rightarrow$ This is a consequence of non-abelian statistics

abelian $\Leftrightarrow$ swapping position of excitations has no effect on state

non-abelian $\Leftrightarrow$ swapping position of excitations changes the wave function